DOMAIN DEFORMATIONS AND EIGENVALUES OF THE DIRICHLET LAPLACIAN IN A RIEMANNIAN MANIFOLD

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Abstract. For any bounded regular domain Ω of a real analytic Riemannian manifold M, we denote by $\lambda_k(\Omega)$ the k-th eigenvalue of the Dirichlet Laplacian of Ω. In this paper, we consider λ_k and as a functional upon the set of domains of fixed volume in M. We introduce and investigate a natural notion of critical domain for this functional. In particular, we obtain necessary and sufficient conditions for a domain to be critical, locally minimizing or locally maximizing for λ_k. These results rely on Hadamard type variational formulae that we establish in this general setting.

As an application, we obtain a characterization of critical domains of the trace of the heat kernel under Dirichlet boundary conditions.

1. Introduction

Isoperimetric eigenvalue problems constitute one of the main topics in spectral geometry and shape optimization. Given a Riemannian manifold M, a natural integer k and a positive constant V, the problem is to optimize the k-th eigenvalue of the Dirichlet Laplacian, considered as a functional upon the set of all bounded domains of volume V of M.

The first result in this subject is the famous Faber-Krahn Theorem [14, 20], originally conjectured by Rayleigh, stating that Euclidean balls minimize the first eigenvalue of the Dirichlet Laplacian among all domains of given volume. Extensions of this classical result to higher order eigenvalues, combinations of eigenvalues as well as domains of other Riemannian manifolds or subjected to other types of constraints, have been obtained during the last decades and a very rich literature is devoted to this subject (see for instance [2, 3, 4, 5, 8, 9, 10, 11, 18, 25, 26, 30, 35, 36, 37] and the references therein).

A fundamental tool in the proof of many results concerning the first Dirichlet eigenvalue is the following variation formula, known as Hadamard's formula

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[17, 15, 32, 33]:

$$\frac{d}{d\varepsilon}\lambda_1(\Omega_\varepsilon)\big|_{\varepsilon=0} = -\int_{\partial\Omega_0} v\left(\frac{\partial\phi}{\partial\nu}\right)^2 d\sigma,$$

where $\lambda_1(\Omega_\varepsilon)$ stands for the first Dirichlet eigenvalue of the domain Ω_ε, $\frac{\partial\phi}{\partial\nu}$ denotes the normal derivative of the first normalized eigenfunction ϕ of the Dirichlet Laplacian on Ω_0 and v is the normal displacement of the boundary induced by the deformation. This formula shows that a necessary and sufficient condition for a domain $\Omega \subset \mathbb{R}^n$ to be critical for the Dirichlet first eigenvalue functional under fixed volume variations, is that its first Dirichlet eigenfunctions are solutions of the following overdetermined problem:

$$\begin{cases} \Delta\phi = \lambda_1(\Omega)\phi \text{ in } \Omega \\ \\ \phi = 0 \quad \text{on } \partial\Omega \\ \\ |\frac{\partial\phi}{\partial\nu}| = c \quad \text{on } \partial\Omega, \end{cases}$$

for some constant c. Since a first Dirichlet eigenfunction does not change sign in Ω, it follows from the well known symmetry result of Serrin [34] that ϕ is radial and Ω is a round ball. Therefore, Euclidean balls are the only critical domains of the Dirichlet first eigenvalue functional under fixed volume deformations.

Notice that Hadamard's formula remains valid for any higher order eigenvalue λ_k as far as $\lambda_k(\Omega)$ is simple. Nevertheless, when $\lambda_k(\Omega)$ is degenerate, a differentiability problem arises and our first aim in this paper (see Section 3) is to introduce, in spite of this non-differentiability problem, a natural and simple notion of critical domain.

Indeed, using perturbation theory of unbounded self-adjoint operators in Hilbert spaces, we will see that, for any deformation Ω_ε, analytic in ε, of a domain Ω of a real analytic Riemannian manifold M, and any natural integer k, the function $\varepsilon \mapsto \lambda_k(\Omega_\varepsilon)$ admits a left sided and a right sided derivatives at $\varepsilon = 0$. Of course, when Ω is a local extremum of λ_k, these derivatives have opposite signs. This suggests us to define critical domains of λ_k to be the domains Ω such that, for any analytic volume-preserving deformation Ω_ε of Ω, the right sided and the left sided derivatives of $\lambda_k(\Omega_\varepsilon)$ at $\varepsilon = 0$ have opposite signs. That is,

$$\frac{d}{d\varepsilon}\lambda_k(\Omega_\varepsilon)\big|_{\varepsilon=0^+} \times \frac{d}{d\varepsilon}\lambda_k(\Omega_\varepsilon)\big|_{\varepsilon=0^-} \leq 0.$$

which means that $\lambda_k(\Omega_\varepsilon) \leq \lambda_k(\Omega) + o(\varepsilon)$ or $\lambda_k(\Omega_\varepsilon) \geq \lambda_k(\Omega_\varepsilon) + o(\varepsilon)$ as $\varepsilon \to 0$.

After giving, in Section 2, a general Hadamard type variation formula, we derive, in Section 3, necessary and sufficient conditions for a domain Ω of the Riemannian manifold M to be critical for the k-th Dirichlet eigenvalue functional under volume-preserving domain deformations.

For instance, we show that (Theorem 3.3) if Ω is a critical domain of the k-th Dirichlet eigenvalue under volume-preserving domain deformations, then there exists a family of eigenfunctions $\phi_1, \dots, \phi_m$ satisfying the following system:

$$
\begin{cases}
\Delta\phi_i = \lambda_k(\Omega)\,\phi_i \text{ in } \Omega,\ \ \forall i \leq m, \\[2ex]
\phi_i = 0 \text{ on } \partial\Omega,\ \forall i \leq m, \\[2ex]
\sum_{i=1}^{m} \left(\frac{\partial\phi_i}{\partial\nu}\right)^2 = 1 \text{ on } \partial\Omega.
\end{cases} \tag{1}
$$

Moreover, this necessary condition is also sufficient when either $\lambda_k(\Omega) > \lambda_{k-1}(\Omega)$ or $\lambda_k(\Omega) < \lambda_{k+1}(\Omega)$, which means that $\lambda_k(\Omega)$ corresponds to the first one or the last one in a cluster of equal eigenvalues. On the other hand, we prove that if $\lambda_k(\Omega) > \lambda_{k-1}(\Omega)$ (resp. $\lambda_k(\Omega) < \lambda_{k+1}(\Omega)$) and if $\Omega \subset M$ is a local minimizer (resp. maximizer) of the k-th Dirichlet eigenvalue functional under volume-preserving domain deformations, then $\lambda_k(\Omega)$ is simple and the absolute value of the normal derivative of its corresponding eigenfunction is constant along the boundary $\partial\Omega$ (see Theorem 3.1).

The last section deals with the trace of the heat kernel under Dirichlet boundary conditions defined for a domain $\Omega \subset M$ by

$$Y_\Omega(t) = \int_\Omega H(t,x,x)v_g = \sum_{k\geq 1} e^{-\lambda_k(\Omega)t},$$

where H is the fundamental solution of the heat equation in Ω under Dirichlet boundary conditions. Indeed, Luttinger [23] proved an isoperimetric Faber-Krahn like result for $Y(t)$ considered as a functional upon the set of bounded Euclidean domains, that is, for any bounded domain $\Omega \subset \mathbb{R}^n$ and any $t > 0$, one has $Y_\Omega(t) \leq Y_{\Omega^*}(t)$, where Ω^* is an Euclidean ball whose volume is equal to that of Ω.

For any smooth deformation Ω_ε of Ω, the corresponding heat trace function $Y_\varepsilon(t)$ is always differentiable w.r.t. ε and the domain Ω will be said critical for the trace of the heat kernel under the Dirichlet boundary condition at time t if, for any volume-preserving deformation Ω_ε of Ω, we have

$$\frac{d}{d\varepsilon} Y_\varepsilon(t)\Big|_{\varepsilon=0} = 0$$

After giving the first variation formula for this functional (Theorem 4.1), we show that a necessary and sufficient condition for a domain Ω to be critical for the trace of the heat kernel under Dirichlet boundary condition at time t

is that the Laplacian of the function $x \mapsto H(t, x, x)$ must be constant along the boundary $\partial\Omega$ (Corollary 4.1).

Using Minakshisundaram-Pleijel asymptotic expansion of $Y(t)$, one can derive necessary conditions for a domain to be critical for the trace of the heat kernel under Dirichlet boundary condition at any time $t > 0$. For instance, we show that the boundary of such a domain necessarily has constant mean curvature (Theorem 4.2).

Thanks to Alexandrov type results (see [1, 24]), one deduces that when the ambient space M is Euclidean, Hyperbolic or a standard hemisphere, then geodesic balls are the only critical domains of the trace of the heat kernel under Dirichlet boundary condition at any time $t > 0$ (Corollary 4.3).

2. Hadamard type variation formulae

Let Ω be a regular bounded domain of a Riemannian oriented manifold (M, g). We will denote by $\bar{g}$ the metric induced by g on the boundary $\partial\Omega$ of Ω. Let us start with the following general formula.

Proposition 2.1. *Let (g_ε) be a differentiable variation of the metric g. Let $\phi_\varepsilon \subset \mathcal{C}^\infty(\Omega)$ be a differentiable family of functions and Λ_ε a differentiable family of real numbers such that, $\forall\varepsilon$, $||\phi_\varepsilon||_{L^2(\Omega, g_\varepsilon)} = 1$ and*

$$\begin{cases} \Delta_{g_\varepsilon}\phi_\varepsilon = \Lambda_\varepsilon\phi_\varepsilon \ \ \text{in } \Omega \\ \\ \phi_\varepsilon = 0 \ \ \text{on } \partial\Omega. \end{cases}$$

Then,

$$\begin{aligned} \frac{d}{d\varepsilon}\Lambda_\varepsilon\big|_{\varepsilon=0} &= \int_\Omega \phi_0 \Delta' \phi_0 v_g \\ &= -\int_\Omega \langle d\phi_0 \otimes d\phi_0 + \frac{1}{4}\Delta\phi_0^2 \ g, h\rangle v_g, \end{aligned}$$

where $h := \frac{d}{d\varepsilon} g_\varepsilon\big|_{\varepsilon=0}$, $\Delta' := \frac{d}{d\varepsilon}\Delta_{g_\varepsilon}\big|_{\varepsilon=0}$ and $\langle , \rangle$ is the inner product induced by g on the space of covariant tensors.

Proof. For simplicity, let us introduce the following notations: $\lambda := \Lambda_0$, $\phi := \phi_0$, $\phi' := \frac{d}{d\varepsilon}\phi_\varepsilon\big|_{\varepsilon=0}$ and $\Lambda' := \frac{d}{d\varepsilon}\Lambda_\varepsilon\big|_{\varepsilon=0}$.

Differentiating the two sides of the equality $\Delta_{g_\varepsilon}\phi_\varepsilon = \Lambda_\varepsilon\phi_\varepsilon$ we obtain

$$\Delta'\phi + \Delta\phi' = \Lambda'\phi + \Lambda\phi'.$$

After multiplication by ϕ and integration we get

$$\int_\Omega \phi\Delta'\phi v_g + \int_\Omega \phi\Delta\phi' v_g = \Lambda' + \lambda\int_\Omega \phi\phi' v_g.$$

Integration by parts gives

$$\int_\Omega \phi\Delta\phi' v_g = \lambda\int_\Omega \phi\phi' v_g + \int_{\partial\Omega}\left(\frac{\partial\phi}{\partial\nu}\phi' - \frac{\partial\phi'}{\partial\nu}\phi\right)v_{\bar{g}}.$$

Thus,

$$\Lambda' = \int_\Omega \phi\Delta'\phi v_g + \int_{\partial\Omega} (\phi' \frac{\partial\phi}{\partial\nu} - \phi\frac{\partial\phi'}{\partial\nu}) v_{\bar g}.$$

It is clear that the boundary integral in this last equation vanishes (since $\phi_\varepsilon = 0$ on $\partial\Omega$).
In conclusion, we have

$$\Lambda' = \int_\Omega \phi\Delta'\phi v_g. \tag{2}$$

Now, the expression of Δ' is given by (see [4])

$$\Delta'\phi = \langle D^2\phi, h\rangle - \langle d\phi, \delta h + \frac{1}{2} d\tilde h\rangle, \tag{3}$$

where $\tilde h$ is the trace of h w.r.t. g (that is $\tilde h = \langle g, h\rangle$). Integration by parts yields

$$\begin{aligned} \int_\Omega \phi\langle d\phi, \delta h\rangle v_g &= \frac{1}{2}\int_\Omega \langle D^2\phi^2, h\rangle v_g \\ &= \int_\Omega \langle d\phi\otimes d\phi + \phi D^2\phi, h\rangle v_g \end{aligned} \tag{4}$$

and

$$\int_\Omega \phi\langle d\phi, d\tilde h\rangle v_g = \frac{1}{2}\int_\Omega \tilde h\Delta\phi^2 v_g \tag{5}$$

Combining (2), (3), (4) and (5) we obtain

$$\Lambda' = -\int_\Omega \langle d\phi\otimes d\phi + \frac{1}{4}\Delta\phi^2 g, h\rangle v_g$$

which completes the proof of the proposition. □

In the particular case of domain deformations, Proposition 2.1 gives rise to the following variation formulae.

Corollary 2.1. *Let $\Omega_\varepsilon = f_\varepsilon(\Omega)$ be a deformation of Ω. Let $\phi_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon)$ and $\Lambda_\varepsilon \in \mathbf{R}$ be two differentiable curves such that, $\forall\varepsilon$, $||\phi_\varepsilon||_{L^2(\Omega_\varepsilon,g)} = 1$ and*

$$\begin{cases} \Delta\phi_\varepsilon = \Lambda_\varepsilon\phi_\varepsilon \ \text{ in } \Omega_\varepsilon \\ \\ \phi_\varepsilon = 0 \ \text{ on } \partial\Omega_\varepsilon. \end{cases}$$

Then,

$$\frac{d}{d\varepsilon}\Lambda_\varepsilon\big|_{\varepsilon=0} = -\int_{\partial\Omega} v\left(\frac{\partial\phi}{\partial\nu}\right)^2 v_{\bar g},$$

where $\phi = \phi_0$ and $v = g\left(\frac{d}{d\varepsilon} f_\varepsilon\big|_{\varepsilon=0}, \nu\right)$ is the normal component of the variation vector field of the deformation Ω_ε.

Proof. Let us apply Proposition 2.1 with $g_\varepsilon = f_\varepsilon^* g$ and $\bar{\phi}_\varepsilon = \phi_\varepsilon \circ f_\varepsilon$. Indeed, one can easily check that $\|\bar{\phi}_\varepsilon\|_{L^2(\Omega, g_\varepsilon)} = 1$, $\Delta_{g_\varepsilon} \bar{\phi}_\varepsilon = \Lambda_\varepsilon \bar{\phi}_\varepsilon$ in Ω and $\bar{\phi}_\varepsilon = 0$ on $\partial\Omega$. Hence,

$$\frac{d}{d\varepsilon} \Lambda_\varepsilon\big|_{\varepsilon=0} = -\int_\Omega \langle d\phi \otimes d\phi + \frac{1}{4}\Delta\phi^2 \ g, h\rangle v_g \tag{6}$$

with $\phi := \phi_0 = \bar{\phi}_0$ and $h = \frac{d}{d\varepsilon} f_\varepsilon^* g\big|_{\varepsilon=0} = \mathcal{L}_V g$, where $\mathcal{L}_V g$ is the Lie derivative of g w.r.t. the vector field $V = \frac{d}{d\varepsilon} f_\varepsilon\big|_{\varepsilon=0}$.

Using the expression of $\mathcal{L}_V g$ in terms of the covariant derivative ∇V of V and integrating by parts, we obtain

$$\begin{aligned}
\int_\Omega & \langle d\phi \otimes d\phi, \mathcal{L}_V g\rangle v_g = \int_\Omega \mathcal{L}_V g(\nabla\phi, \nabla\phi) v_g = 2\int_\Omega \langle \nabla_{\nabla\phi} V, \nabla\phi\rangle v_g \\
= & \int_\Omega \mathrm{div}(\langle V, \nabla\phi\rangle \nabla\phi) v_g + 2\int_\Omega \langle V, \nabla\phi\rangle \Delta\phi v_g - 2\int_\Omega D^2\phi(V, \nabla\phi) v_g \\
= & \ 2\int_{\partial\Omega} \langle V, \nabla\phi\rangle \frac{\partial\phi}{\partial\nu} \ v_{\bar{g}} + \lambda \int_\Omega \langle V, \nabla\phi^2\rangle v_g - 2\int_\Omega D^2\phi(V, \nabla\phi) v_g,
\end{aligned}$$

with $\lambda := \Lambda_0$, and

$$\begin{aligned}
\frac{1}{4}\int_\Omega \Delta\phi^2 \langle g, \mathcal{L}_V g\rangle v_g &= \frac{1}{2}\int_\Omega \Delta\phi^2 \mathrm{div} V v_g \\
&= \lambda \int_\Omega \phi^2 \mathrm{div} V v_g - \int_\Omega |\nabla\phi|^2 \mathrm{div} V v_g \\
&= \int_\Omega (-\lambda\langle V, \nabla\phi^2\rangle + 2D^2\phi(V, \nabla\phi)) v_g \\
&+ \int_{\partial\Omega} (\lambda\phi^2 - |\nabla\phi|^2)\langle V, \nu\rangle v_{\bar{g}}.
\end{aligned}$$

Replacing in (6), we get

$$\frac{d}{d\varepsilon}\Lambda_\varepsilon\big|_{\varepsilon=0} = \int_{\partial\Omega} \{-2\langle V, \nabla\phi\rangle \frac{\partial\phi}{\partial\nu} + \langle V, \nu\rangle |\nabla\phi|^2 - \lambda\langle V, \nu\rangle \phi^2\} v_{\bar{g}}.$$

Since ϕ is identically zero on the boundary, we have at any point of $\partial\Omega$, $\nabla\phi = \frac{\partial\phi}{\partial\nu}\nu$. In particular, $|\nabla\phi|^2 = \left(\frac{\partial\phi}{\partial\nu}\right)^2$ and $\langle V, \nabla\phi\rangle = \langle V, \nu\rangle \frac{\partial\phi}{\partial\nu} = v\frac{\partial\phi}{\partial\nu}$. Thus,

$$\frac{d}{d\varepsilon} \ \Lambda_\varepsilon\big|_{\varepsilon=0} = -\int_{\partial\Omega} v \left(\frac{\partial\phi}{\partial\nu}\right)^2 v_{\bar{g}}.$$

□

3. Critical domains

Throughout this section, the ambient Riemannian manifold (M, g) is assumed to be real analytic.

3.1. **Preliminary results and definitions.** Let Ω be a regular bounded domain of a Riemannian manifold (M, g). An analytic deformation (Ω_ε) of Ω is given by an analytic 1-parameter family of diffeomorphisms $f_\varepsilon : \Omega \to \Omega_\varepsilon$ such that $f_\varepsilon(\partial\Omega) = \partial\Omega_\varepsilon$ and $f_0 = Id$. Such a deformation is called volume-preserving if the Riemannian volume of Ω_ε w.r.t the metric g does not depend on ε.

The spectrum of the Laplace operator Δ_g under the Dirichlet boundary condition will be denoted

$$Sp_{_D}\ (\Delta_g, \Omega_\varepsilon) = \{\ \lambda_{1,\varepsilon}\ < \lambda_{2,\varepsilon}\ \leq \cdots \leq\ \lambda_{k,\varepsilon}\quad \uparrow + \infty\ \}$$

The functions $\varepsilon \mapsto \lambda_{k,\varepsilon}$ is continuous but not differentiable in general, excepting $\lambda_{1,\varepsilon}$ which is always differentiable since it is simple. However, as we will see hereafter, the general perturbation theory of unbounded self-adjoint operators enables us to show that the function $\lambda_{k,\varepsilon}$ admits a right sided and a left sided derivatives at $\varepsilon = 0$. In all the sequel, a family of functions $\phi_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon)$ will be said differentiable (resp. analytic) w.r.t ε, if that is the case for $\phi_\varepsilon \circ f_\varepsilon \in \mathcal{C}^\infty(\Omega)$.

Lemma 3.1. *Let $\lambda \in Sp_{_D}(\Delta_g, \Omega)$ be an eigenvalue of multiplicity p of the Dirichlet Laplacian in Ω. For any analytic deformation Ω_ε of Ω, there exist p families $(\Lambda_{i,\varepsilon})_{i\leq p}$ of real numbers and p families $(\phi_{i,\varepsilon})_{i\leq p} \subset \mathcal{C}^\infty(\Omega_\varepsilon)$ of functions, depending analytically on ε and satisfying, $\forall\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and $\forall i \in \{1, \ldots, p\}$,*

(a) $\Lambda_{i,0} = \lambda$

(b) *the family $\{\phi_{1,\varepsilon}, \cdots, \phi_{p,\varepsilon}\}$ is orthonormal in $L^2(\Omega_\varepsilon, g)$*

(c) $\begin{cases} \Delta\ \phi_{i,\varepsilon}\ = \Lambda_{i,\varepsilon}\phi_{i,\varepsilon}\ \text{in}\ \Omega_\varepsilon \\ \\ \phi_{i,\varepsilon} = 0\ \text{on}\ \partial\Omega_\varepsilon \end{cases}$

The proof is based on perturbation theory of unbounded self-adjoint operators in Hilbert spaces. Results concerning the differentiability of eigenvalues and eigenvectors have been first obtained by Rellich [31] and then by Kato [19] in the analytic case. Many results were also obtained under weaker differentiability conditions (see for instance [21, 22] for recent contributions in this subject). Nevertheless, even a smooth curve $\varepsilon \mapsto P_\varepsilon$ of self-adjoint operators may lead to noncontinuous eigenvectors w.r.t. ε (see Rellich's example [19, chap II, Example 5.3]). Since we need to differentiate eigenvectors w.r.t. ε, we imposed analyticity assumptions in order to get analytic curves of operators.

Proof of Lemma 3.1. In order to get into the framework of perturbation theory, we first need to modify our operators so that they all have the same domain. Indeed, for any ε we set $g_\varepsilon = f_\varepsilon^* g$ and denote Δ_ε the Laplace operator of (Ω, g_ε). Clearly, we have

$$Sp_{_D}(\Delta_g, \Omega_\varepsilon) = Sp_{_D}(\Delta_\varepsilon, \Omega).$$

Notice that since f_ε depends analytically on ε and that g is real analytic, the curves $\varepsilon \mapsto g_\varepsilon$ and, hence, $\varepsilon \mapsto \Delta_\varepsilon$ are analytic w.r.t. ε.

The operator Δ_ε is symmetric w.r.t. the inner product in $L^2(\Omega, g_\varepsilon)$, but not necessarily w.r.t. the inner product in $L^2(\Omega, g)$. Therefore, we need to introduce a conjugation as follows. Let $U_\varepsilon : L^2(\Omega, g) \to L^2(\Omega, g_\varepsilon)$ be the unitary isomorphism given by

$$U_\varepsilon : v \mapsto \left(\frac{|g|}{|g_\varepsilon|}\right)^{\frac{1}{4}} v,$$

where $|g| = \det(g_{ij})$ is the determinant of the matrix (g_{ij}) of the components of g in a local coordinate system. We define the operator P_ε to be

$$P_\varepsilon = U_\varepsilon^{-1} \circ \Delta_\varepsilon \circ U_\varepsilon.$$

Therefore, we have $Sp_{_D}(P_\varepsilon, \Omega) = Sp_{_D}(\Delta_\varepsilon, \Omega)$ and, if $v_\varepsilon \in \mathcal{C}^\infty(\Omega)$ is an eigenfunction of P_ε, then $\phi_\varepsilon = U_\varepsilon(v_\varepsilon) \circ f_\varepsilon^{-1} \in \mathcal{C}^\infty(\Omega_\varepsilon)$ is an eigenfunction of Δ_g with the same eigenvalue. Again, since $\forall\varepsilon$, (M, g_ε) is real analytic, the curves $\varepsilon \mapsto U_\varepsilon$ and $\varepsilon \mapsto P_\varepsilon$ are analytic. The result of the lemma then follows from the Rellich-Kato theory applied to $\varepsilon \mapsto P_\varepsilon$.

□

Now, let us fix a positive integer k and let $\Lambda_{1,\varepsilon}, \dots, \Lambda_{p,\varepsilon}$ be the family of eigenvalues associated with λ_k by Lemma 3.1. Using the continuity of $\lambda_{k,\varepsilon}$ and the analyticity of $\Lambda_{i,\varepsilon}$ w.r.t. ε, we can easily see that there exist two integers $i \le p$ and $j \le p$ such that

$$\lambda_{k,\varepsilon} = \begin{cases} \Lambda_{i,\varepsilon} \text{ if } \varepsilon \le 0 \\ \\ \Lambda_{j,\varepsilon} \text{ if } \ \varepsilon \ge 0. \end{cases}$$

Hence, $\lambda_{k,\varepsilon}$ admits a left sided and a right sided derivatives with

$$\frac{d}{d\varepsilon}\lambda_{k,\varepsilon}\big|_{\varepsilon=0^+} \quad = \frac{d}{d\varepsilon}\Lambda_{j,\varepsilon}\big|_{\varepsilon=0}$$

and

$$\frac{d}{d\varepsilon}\lambda_{k,\varepsilon}\big|_{\varepsilon=0^-} \quad = \frac{d}{d\varepsilon}\Lambda_{i,\varepsilon}\big|_{\varepsilon=0}.$$

Definition 3.1. *The domain Ω is said to be "critical" for the k-th eigenvalue of Dirichlet problem if, for any analytic volume-preserving deformation Ω_ε of Ω, the right sided and the left sided derivatives of $\lambda_{k,\varepsilon}$ at $\varepsilon = 0$ have opposite signs. That is,*

$$\frac{d}{d\varepsilon}\lambda_{k,\varepsilon}\big|_{\varepsilon=0^+} \times \frac{d}{d\varepsilon}\lambda_{k,\varepsilon}\big|_{\varepsilon=0^-} \le 0.$$

It is easy to see that

$$\frac{d}{d\varepsilon}\lambda_{k,\varepsilon}\big|_{\varepsilon=0^+} \leq 0 \leq \frac{d}{d\varepsilon}\lambda_{k,\varepsilon}\big|_{\varepsilon=0^-} \Longleftrightarrow \lambda_{k,\varepsilon} \leq \lambda_{k,0} + o(\varepsilon)$$

and

$$\frac{d}{d\varepsilon}\lambda_{k,\varepsilon}\big|_{\varepsilon=0^-} \leq 0 \leq \frac{d}{d\varepsilon}\lambda_{k,\varepsilon}\big|_{\varepsilon=0^+} \Longleftrightarrow \lambda_{k,\varepsilon} \geq \lambda_{k,0} + o(\varepsilon).$$

Therefore, the domain Ω is critical for the k-th eigenvalue of Dirichlet problem if and only if one of the following inequalities holds:

$$\lambda_{k,\varepsilon} \leq \lambda_{k,0} + o(\varepsilon)$$
$$\lambda_{k,\varepsilon} \geq \lambda_{k,0} + o(\varepsilon).$$

Remark 3.1. *Suppose that for an integer k we have $\lambda_k < \lambda_{k+1}$, then, for sufficiently small ε, we will have $\lambda_{k,\varepsilon} = \max_{i\leq p} \Lambda_{i,\varepsilon}$, where $\Lambda_{1,\varepsilon}, \ldots, \Lambda_{p,\varepsilon}$ are the eigenvalues associated to λ_k by Lemma 3.1 (indeed, $\Lambda_{i,0} = \lambda_k < \lambda_{k+1}$ for any $1 \leq i \leq p$). Hence, $\frac{d}{d\varepsilon}\ \lambda_{k,\varepsilon}|_{\varepsilon=0^-} \leq \frac{d}{d\varepsilon}\lambda_{k,\varepsilon}|_{\varepsilon=0^+}$. In particular, Ω is critical for the functional $\Omega \mapsto \lambda_k(\Omega)$ if and only if $\frac{d}{d\varepsilon}\lambda_{k,\varepsilon}|_{\varepsilon=0^-} \leq 0 \leq \frac{d}{d\varepsilon}\lambda_{k,\varepsilon}|_{\varepsilon=0^+}$ (or, equivalently, $\lambda_{k,\varepsilon} \leq \lambda_{k,0} + o(\varepsilon)$).*

Similarly, if $\lambda_{k-1} < \lambda_k$, then, for sufficiently small ε, $\lambda_{k,\varepsilon} = \min_{i\leq p} \Lambda_{i,\varepsilon}$ and $\frac{d}{d\varepsilon}\lambda_{k,\varepsilon}|_{\varepsilon=0^+} \leq \frac{d}{d\varepsilon}\ \lambda_{k,\varepsilon}|_{\varepsilon=0^-}$.

Lemma 3.2. *Let $\lambda \in Sp_{D}(\Delta_g, \Omega)$ be an eigenvalue of multiplicity p of the Dirichlet Laplacian in Ω and let us denote by E_λ the corresponding eigenspace. Let $\Omega_\varepsilon = f_\varepsilon(\Omega)$ be an analytic deformation of Ω and let $(\Lambda_{i,\varepsilon})_{i\leq p}$ and $(\phi_{i,\varepsilon})_{i\leq p} \subset \mathcal{C}^\infty(\Omega_\varepsilon)$ be as in Lemma 3.1. Then $\Lambda_1' := \frac{d}{d\varepsilon}\Lambda_{1,\varepsilon}\big|_{\varepsilon=0}, \cdots, \Lambda_p' := \frac{d}{d\varepsilon}\Lambda_{p,\varepsilon}\big|_{\varepsilon=0}$ are the eigenvalues of the quadratic form q_v defined on the space $E_\lambda \subset L^2(\Omega, g)$ by*

$$q_v(\phi) = -\int_{\partial\Omega} v \left(\frac{\partial\phi}{\partial\nu}\right)^2 v_{\bar{g}},$$

where $v = g\left(\frac{d}{d\varepsilon}f_\varepsilon\big|_{\varepsilon=0}, \nu\right)$. Moreover, the L^2-orthonormal basis $\phi_{1,0}, \cdots, \phi_{p,0}$ diagonalizes q_v on E_λ.

Proof. For simplicity, we set $g_\varepsilon := f_\varepsilon^* g$, $\Delta' := \frac{d}{d\varepsilon}\Delta_{g_\varepsilon}\big|_{\varepsilon=0}$, $\Lambda_i := \Lambda_{i,0}$, $\phi_i := \phi_{i,0}$ and $\Lambda_i' := \frac{d}{d\varepsilon}\Lambda_{i,\varepsilon}\big|_{\varepsilon=0}$. From $\Delta_{g_\varepsilon}(\phi_{i,\varepsilon}) = \Lambda_{i,\varepsilon}(\phi_{i,\varepsilon})$, we deduce

$$\Delta'\phi_i + \Delta\phi_i' = \Lambda_i'\phi_i + \Lambda_i\phi_i'$$

We multiply by ϕ_j and integrate to get

$$\int_\Omega \phi_j \Delta'\phi_i v_g + \int_\Omega \phi_j \Delta\phi_i' v_g = \Lambda_i' \int_\Omega \phi_i\phi_j v_g + \lambda \int_\Omega \phi_i\phi_j' v_g.$$

Integration by parts gives (since $\phi_j = \phi_i^{'} = 0$ on $\partial\Omega$))

$$\int_\Omega \phi_j \Delta \phi_i' v_g = \lambda \int_\Omega \phi_i \phi_j' v_g.$$

Hence,

$$\int_\Omega \phi_j \Delta' \phi_i v_g = \Lambda_i' \int_\Omega \phi_i \phi_j v_g.$$

Therefore

$$\int_\Omega \phi_j \Delta' \phi_i v_g = \Lambda_i' \int_\Omega \phi_i \phi_j v_g.$$

It follows that the L^2-orthonormal basis $\phi_1, \cdots, \phi_p$ diagonalizes the quadratic form $\phi \to \int_\Omega \phi \Delta' \phi v_g$ on E_λ, the corresponding eigenvalues being $\Lambda_1', \cdots, \Lambda_p'$. As we have seen in the proof of the corollary 2.1, this last quadratic form coincides with q_v on E_λ. $\square$

Any *volume-preserving* deformation $\Omega_\varepsilon = f_\varepsilon(\Omega)$ induces a function $v := g(\frac{d}{d\varepsilon} f_\varepsilon\big|_{\varepsilon=0}, \nu)$ on $\partial\Omega$ satisfying $\int_{\partial\Omega} v\, v_{\bar g} = 0$ (indeed, this last integral is equal up to a constant to $\frac{d}{d\varepsilon} vol(\Omega_\varepsilon)\big|_{\varepsilon=0}$). In all the sequel, we will denote by $\mathcal{A}_0(\partial\Omega)$ the set of regular functions on $\partial\Omega$ such that $\int_{\partial\Omega} v\, v_{\bar g} = 0$. The following elementary lemma will be useful in the proof of our main results.

Lemma 3.3. *Let $v \in \mathcal{A}_0(\partial\Omega)$. Then there exists an analytic volume-preserving deformation $\Omega_\varepsilon = f_\varepsilon(\Omega)$ so that $v = g(\frac{d}{d\varepsilon} f_\varepsilon\big|_{\varepsilon=0}, \nu)$.*

Proof. Let $U \subset M$ be an open neighborhood of $\bar\Omega$ and let $\tilde v$ and $\tilde \nu$ be smooth extensions to U of v and ν respectively. For ε sufficiently small, the map $\varphi_\varepsilon(x) = \exp_x \varepsilon\, \tilde v(x)\, \tilde\nu(x)$ is a diffeomorphism from Ω to $\varphi_\varepsilon(\Omega)$. Moreover, since (M, g) is real analytic, the curve $\varepsilon \to \varphi_\epsilon$ is analytic w.r.t. ε. The deformation $\varphi_\varepsilon(\Omega)$ is not necessarily volume-preserving. However, let X be any analytic vectorfield on U such that $\int_\Omega \mathrm{div} X v_g \neq 0$ and denote by $(\gamma_t)_t$ the associated 1-parameter local group of diffeomorphisms. The function $(t, \varepsilon) \mapsto F(t, \varepsilon) = vol(\gamma_t \circ \varphi_\varepsilon(\Omega))$ satisfies $\frac{\partial}{\partial t} F(0,0) = \int_\Omega \mathrm{div} X v_g \neq 0$. Applying the implicit function theorem in the analytic setting, we get the existence of a function $t(\varepsilon)$ depending analytically on $\varepsilon \in (-\eta, \eta)$, for some $\eta > 0$ sufficiently small, such that $F(t(\varepsilon), \varepsilon) = F(0, 0)$, $\forall \varepsilon \in (-\eta, \eta)$. The deformation $f_\varepsilon = \gamma_{t(\varepsilon)} \circ \varphi_\varepsilon$ is clearly analytic and volume-preserving. Moreover, one has

$$t'(0) = -\frac{\frac{d}{d\varepsilon} vol(\varphi_\varepsilon(\Omega))\big|_{\varepsilon=0}}{\frac{d}{dt} vol(\gamma_t(\Omega))\big|_{t=0}} = -\frac{\int_\Omega \mathrm{div} \tilde v \tilde \nu\, v_g}{\int_\Omega \mathrm{div} X v_g} = -\frac{\int_{\partial\Omega} v\, v_{\bar g}}{\int_{\partial\Omega} \langle X, \nu\rangle\, v_{\bar g}} = 0.$$

Therefore, $\forall x \in \partial\Omega$,

$$\frac{d}{d\varepsilon} f_\varepsilon(x)\big|_{\varepsilon=0} = t'(0) X(x) + \frac{d\varphi_\epsilon(x)}{d\varepsilon}\big|_{\varepsilon=0} = v(x)\nu(x).$$

$\square$

3.2. **Critical domains for the k-th eigenvalue of the Dirichlet Laplacian.** In all the sequel, we will denote by λ_k the k-th eigenvalue of the Dirichlet problem in Ω and by E_k the corresponding eigenspace.

In the following results, a special role is played by the eigenvalues λ_k satisfying $\lambda_k > \lambda_{k-1}$ or $\lambda_k < \lambda_{k+1}$. This means that the index k is the lowest or the highest one among all the indices corresponding to the same eigenvalue. Let us start with the following necessary condition to be satisfied by a locally minimizing or locally maximizing domain. Here, a local minimizer (resp. maximizer) for the k-th eigenvalue of the Dirichlet Laplacian is a domain Ω such that, for any volume-preserving deformation Ω_ε, the function $\varepsilon \mapsto \lambda_{k,\varepsilon}$ admits a local minimum (resp. maximum) at $\varepsilon = 0$.

Theorem 3.1. *Let k be a natural integer such that $\lambda_k > \lambda_{k-1}$ (resp. $\lambda_k < \lambda_{k+1}$) and assume that Ω is a local minimizer (resp. local maximizer) for the k-th eigenvalue of the Dirichlet Laplacian. Then λ_k is simple and the absolute value of the normal derivative of its corresponding eigenfunction is constant on $\partial\Omega$. That is, there exists a unique (up to sign) function ϕ satisfying*

$$\begin{cases} \Delta\phi = \lambda_k\phi \ \textit{in } \Omega \\ \\ \phi = 0 \quad \textit{on } \partial\Omega \\ \\ \left|\frac{\partial\phi}{\partial\nu}\right| = 1 \quad \textit{on } \partial\Omega. \end{cases}$$

Proof. Suppose that $\lambda_k > \lambda_{k-1}$ and let $\Omega_\varepsilon = f_\varepsilon(\Omega)$ be a volume preserving analytic deformation of Ω. Let $(\Lambda_{i,\varepsilon})_{i\le p}$ and $(\phi_{i,\varepsilon})_{i\le p}$ be families of eigenvalues and eigenfunctions associated to λ_k according to Lemma 3.1. Since $\Lambda_{i,0} = \lambda_k > \lambda_{k-1}$, we have, for sufficiently small ε, for continuity reasons,

$$\Lambda_{i,\varepsilon} > \lambda_{k-1,\varepsilon}.$$

Hence,

$$\Lambda_{i,\varepsilon} \ge \lambda_{k,\varepsilon}.$$

As the function $\varepsilon \mapsto \lambda_{k,\varepsilon}$ admits a local minimum at $\varepsilon = 0$ with $\Lambda_{i,0} = \lambda_{k,0} = \lambda_k$, it follows that the differentiable function $\varepsilon \mapsto \Lambda_{i,\varepsilon}$ achieves a local minimum at $\varepsilon = 0$ and that $\frac{d}{d\varepsilon}\Lambda_{i,\varepsilon}\big|_{\varepsilon=0} = 0$. Applying Lemma 3.2, we deduce that the quadratic form q_v is identically zero on the eigenspace E_k, where $v = g(\frac{d}{d\varepsilon}f_\varepsilon\big|_{\varepsilon=0}, \nu)$. The volume-preserving deformation being arbitrary, it follows that the form q_v vanishes on E_k for any $v \in \mathcal{A}_0(\partial\Omega)$ (Lemma 3.3). Therefore, $\forall\phi \in E_k$ and $\forall v \in \mathcal{A}_0(\partial\Omega)$, we have $\int_{\partial\Omega} v \left(\frac{\partial\phi}{\partial\nu}\right)^2 v_{\bar{g}} = 0$, which implies that $\frac{\partial\phi}{\partial\nu}$ is locally constant on $\partial\Omega$ for any $\phi \in E_k$. Now, if ϕ_1 and ϕ_2 are two eigenfunctions in E_k, one can find a linear combination $\phi = \alpha\phi_1 + \beta\phi_2$ so that $\frac{\partial\phi}{\partial\nu}$ vanishes on at least one connected component of $\partial\Omega$. We apply Holmgren uniqueness theorem (see for instance [27, Theorem 2, p. 42], and recall that (M,g) is assumed to be real analytic) to deduce that ϕ is identically zero in Ω and that λ_k is simple.

To finish the proof, we must show that, $\forall \phi \in E_k$, $|\frac{\partial \phi}{\partial \nu}|$ takes the same constant value on all the components of $\partial\Omega$. Indeed, let Σ_1 and Σ_2 be two distinct connected components of $\partial\Omega$ and let $v \in \mathcal{A}_0(\partial\Omega)$ be the function given by $v = vol(\Sigma_2)$ on Σ_1, $v = -vol(\Sigma_1)$ on Σ_2 and $v = 0$ on the other components. Then the condition $\int_{\partial\Omega} v \left(\frac{\partial\phi}{\partial\nu}\right)^2 v_{\bar g} = 0$ implies that $\left(\frac{\partial\phi}{\partial\nu}\right)^2 |_{\Sigma_1} = \left(\frac{\partial\phi}{\partial\nu}\right)^2 |_{\Sigma_2}$.

Of course, the same arguments work in the case $\lambda_k < \lambda_{k+1}$. □

The criticality of the domain Ω for the k-th eigenvalue of Dirichlet Laplacian is closely related to the definiteness of the quadratic forms q_v introduced in Lemma 3.2 above, on the eigenspace E_k. Indeed, we have the following

Theorem 3.2. *Let k be any natural integer.*

1. *If Ω is a critical domain for the k-th eigenvalue of the Dirichlet Laplacian, then, $\forall v \in \mathcal{A}_0(\partial\Omega)$, the quadratic form $q_v(\phi) = -\int_{\partial\Omega} v \left(\frac{\partial\phi}{\partial\nu}\right)^2 v_{\bar g}$ is not definite on E_k.*
2. *Assume that $\lambda_k > \lambda_{k-1}$ or $\lambda_k < \lambda_{k+1}$ and that $\forall v \in \mathcal{A}_0(\partial\Omega)$, the quadratic form $q_v(\phi) = -\int_{\partial\Omega} v \left(\frac{\partial\phi}{\partial\nu}\right)^2 v_{\bar g}$ is not definite on E_k, then Ω is a critical domain for the k-th eigenvalue of the Dirichlet Laplacian.*

Proof. (1) Consider a function $v \in \mathcal{A}_0(\partial\Omega)$ and let $\Omega_\varepsilon = f_\varepsilon(\Omega)$ be an analytic volume-preserving deformation of Ω so that $v := g(\frac{d}{d\varepsilon} f_\varepsilon\big|_{\varepsilon=0}, \nu)$ (Lemma 3.3). Let $(\Lambda_{i,\varepsilon})_{i\le p}$ and $(\phi_{i,\varepsilon})_{i\le p}$ be families of eigenvalues and eigenfunctions associated to λ_k according to Lemma 3.1. As we have seen above, there exists two integers $i \le p$ and $j \le p$ so that $\frac{d}{d\varepsilon}\lambda_{k,\varepsilon}\big|_{\varepsilon=0^-} = \frac{d}{d\varepsilon}\Lambda_{i,\varepsilon}\big|_{\varepsilon=0}$ and $\frac{d}{d\varepsilon}\lambda_{k,\varepsilon}\big|_{\varepsilon=0^+} = \frac{d}{d\varepsilon}\Lambda_{j,\varepsilon}\big|_{\varepsilon=0}$. The criticality of Ω then implies that $\frac{d}{d\varepsilon}\Lambda_{i,\varepsilon}\big|_{\varepsilon=0} \times \frac{d}{d\varepsilon}\Lambda_{j,\varepsilon}\big|_{\varepsilon=0} \le 0$. Applying Lemma 3.2, we deduce that the quadratic form q_v admits both nonnegative and nonpositive eigenvalues on E_k which proves Assertion 1.

(2) Assume that $\lambda_k > \lambda_{k-1}$ and let $\Omega_\varepsilon = f_\varepsilon(\Omega)$ be a volume-preserving deformation of Ω. Let $(\Lambda_{i,\varepsilon})_{i\le p}$ and $(\phi_{i,\varepsilon})_{i\le p}$ be families of eigenvalues and eigenfunctions associated to λ_k according to Lemma 3.1. As we have seen in Remark 3.1, we have, for sufficiently small ε, $\lambda_{k,\varepsilon} = \min_{i\le p} \Lambda_{i,\varepsilon}$. Hence,

$$\frac{d}{d\varepsilon}\lambda_{k,\varepsilon}\big|_{\varepsilon=0^+} = \min_{i\le p} \frac{d}{d\varepsilon}\Lambda_{i,\varepsilon}\big|_{\varepsilon=0}$$

and

$$\frac{d}{d\varepsilon}\ \lambda_{k,\varepsilon}\big|_{\varepsilon=0^-} = \max_{i\le p} \frac{d}{d\varepsilon}\Lambda_{i,\varepsilon}\big|_{\varepsilon=0}.$$

Now, the non definiteness of q_v on E_k means that its smallest eigenvalue is nonpositive and its largest one is nonnegative. According to Lemma 3.2, this implies that $\frac{d}{d\varepsilon}\lambda_{k,\varepsilon}\big|_{\varepsilon=0^+} = \min_{i\leq p} \frac{d}{d\varepsilon}\Lambda_{i,\varepsilon}\big|_{\varepsilon=0} \leq 0$ and $\frac{d}{d\varepsilon}\ \lambda_{k,\varepsilon}\big|_{\varepsilon=0^-} = \max_{i\leq p} \frac{d}{d\varepsilon}\Lambda_{i,\varepsilon}\big|_{\varepsilon=0} \geq 0$ which implies the criticality of the domain Ω.

The case $\lambda_k < \lambda_{k+1}$ can be handled similarly. $\square$

The indefiniteness of q_v for any $v \in \mathcal{A}_0(\partial\Omega)$ can be interpreted intrinsically in the following manner:

Lemma 3.4. *Let k be a natural integer. The two following conditions are equivalent:*

(i) $\forall v \in \mathcal{A}_0(\partial\Omega)$, *the quadratic form q_v is not definite on E_k*

(ii) *there exists a finite family of eigenfunctions $(\phi_i)_{i\leq m} \subset E_k$ satisfying*

$$\sum_{i=1}^{m} \left(\frac{\partial \phi_i}{\partial \nu}\right)^2 = 1 \ \text{ on } \partial\Omega.$$

Proof. To see that (ii) implies (i), it suffices to notice that, for any $v \in \mathcal{A}_0(\partial\Omega)$

$$\sum_{i\leq m} q_v(\phi_i) = -\sum_{i\leq m}\int_{\partial\Omega} v \left(\frac{\partial \phi_i}{\partial \nu}\right)^2 v_{\bar g} = -\int_{\partial\Omega} v v_{\bar g} = 0.$$

Therefore, q_v is not definite on E_k.

The proof of "(i) implies (ii)" uses arguments similar to those used in the case of closed manifolds by Nadirashvili [25] and the authors [12]. Let K be the convex hull of $\left\{\left(\frac{\partial\phi}{\partial\nu}\right)^2, \phi \in E_k\right\}$ in $\mathcal{C}^\infty(\partial\Omega)$. Then, we need to show that the constant function 1 belongs to K.

Let us suppose for a contradiction that $1 \notin K$, then, from the Hahn-Banach theorem (applied in the finite dimensional vector space spanned by K and 1 and endowed with the $L^2(\partial\Omega, \bar g)$ inner product), there exists a function $v \in \mathcal{C}^\infty(\partial\Omega)$ such that $\int_{\partial\Omega} v\ v_{\bar g} > 0$ and, $\forall \phi \in E_k$,

$$\int_{\partial\Omega} v \left(\frac{\partial\phi}{\partial\nu}\right)^2 v_{\bar g} \leq 0.$$

Hence, the zero mean value function $v_o = v - \frac{1}{vol(\partial\Omega)}\int_{\partial\Omega} v\ v_{\bar g}$ satisfies, $\forall \phi \in E_k$,

$$\begin{aligned} q_{v_0}(\phi) &= -\int_{\partial\Omega} v_o \left(\frac{\partial\phi}{\partial\nu}\right)^2 v_{\bar g} \\ &= -\int_{\partial\Omega} v \left(\frac{\partial\phi}{\partial\nu}\right)^2 v_{\bar g} + \frac{1}{vol(\partial\Omega)}\int_{\partial\Omega} v\ v_{\bar g}\int_{\partial\Omega}\left(\frac{\partial\phi}{\partial\nu}\right)^2 v_{\bar g} \\ &\geq \frac{1}{vol(\partial\Omega)}\int_{\partial\Omega} v\ v_{\bar g}\int_{\partial\Omega}\left(\frac{\partial\phi}{\partial\nu}\right)^2 v_{\bar g}, \end{aligned}$$

with $\int_{\partial\Omega}\left(\frac{\partial\phi}{\partial\nu}\right)^2 v_{\bar g} > 0$ for any non trivial Dirichlet eigenfunction ϕ (due to Holmgren uniqueness theorem). In conclusion, the function $v_o \in \mathcal{A}_0(\partial\Omega)$ is such that the quadratic form q_{v_0} is positive definite on E_k, which contradicts Condition (i). □

A consequence of this lemma and Theorem 3.2 is the following:

Theorem 3.3. *Let k be any natural integer.*

(1) *If Ω is a critical domain for the k-th eigenvalue of Dirichlet Laplacian, then there exists a finite family of eigenfunctions $(\phi_i)_{i\le m}\subset E_k$ satisfying $\sum_{i=1}^m\left(\frac{\partial\phi_i}{\partial\nu}\right)^2=1$ on $\partial\Omega$, that is, $(\phi_i)_{i\le m}$ are solutions of the following system*

$$\begin{cases}\Delta\phi_i=\lambda_k\phi_i \text{ in } \Omega, \ \ \forall i\le m,\\ \\ \phi_i=0 \text{ on } \partial\Omega, \ \forall i\le m,\\ \\ \sum_{i=1}^m\left(\frac{\partial\phi_i}{\partial\nu}\right)^2=1 \text{ on } \partial\Omega.\end{cases}$$

(2) *Assume that $\lambda_k>\lambda_{k-1}$ or $\lambda_k<\lambda_{k+1}$ and that there exists a finite family of eigenfunctions $(\phi_i)_{i\le m}\subset E_k$ such that $\sum_{i=1}^m\left(\frac{\partial\phi_i}{\partial\nu}\right)^2$ is constant on $\partial\Omega$, then the domain Ω is critical for the k-th eigenvalue of the Dirichlet Laplacian.*

Corollary 3.1. *Assume that λ_k is simple. The domain Ω is critical for the k-th eigenvalue of the Dirichlet Laplacian if and only if the following overdetermined Pompeiu type system admits a solution*

$$\begin{cases}\Delta\phi=\lambda_k\phi \text{ in } \Omega\\ \\ \phi=0 \quad \text{on } \partial\Omega\\ \\ \left|\frac{\partial\phi}{\partial\nu}\right|=1 \quad \text{on } \partial\Omega.\end{cases}$$

3.3. **Nonexistence of critical domains under metric variations.** In this paragraph, we point out the non consistency of the notion of critical domains w.r.t. metric variations under Dirichlet boundary condition. Indeed, if g_ε is an analytic variation of the metric g, then we can associate to each eigenvalue λ_k of the Dirichlet problem in Ω, analytic families $(\Lambda_{i,\varepsilon})_{i\le p}\subset\mathbb{R}$ and $(\phi_{i,\varepsilon})_{i\le p}\subset\mathcal{C}^\infty(\Omega)$ (where p is the multiplicity of λ_k) satisfying, for sufficiently small ε,

(1) $(\phi_{i,\varepsilon})_{i\le p}$ is $L^2(\Omega,g_\varepsilon)$ orthonormal.
(2) $\forall\ i\in\{1,\dots,p\}$, $\Lambda_{i,o}=\lambda_k$.
(3) $\forall\ i\le p$, $\begin{cases}\Delta_{g_\varepsilon}\phi_{i,\varepsilon}=\Lambda_{i,\varepsilon}\phi_{i,\varepsilon} \text{ in } \Omega\\ \\ \phi_{i,\varepsilon}=0 \text{ on } \partial\Omega\end{cases}$

Therefore $\lambda_{k,\varepsilon}$ admits a left sided and a right sided derivatives at $\varepsilon = 0$, and we can mimic Definition 3.1 to introduce the notion of critical domain for the k-th eigenvalue of Dirichlet problem w.r.t. volume-preserving variations of the metric. Thanks to Proposition 2.1 and using arguments similar to those used above (see also [12, 25]), we can show that, if the domain (Ω, g) is critical for the k-th eigenvalue of Dirichlet problem, then there exists a family of eigenfunctions $\phi_1, \ldots, \phi_m \in E_k$ satisfying

$$\sum_{i=1}^{m} d\phi_i \otimes d\phi_i = g. \tag{7}$$

Now, if we consider only volume-preserving conformal variations g_ε of g (that is $g_\varepsilon = \alpha_\varepsilon \; g$ with $\int_\Omega \alpha_\varepsilon^{\frac{n}{2}} v_g = Vol(\Omega, g)$), then the necessary condition (7) for (Ω, g) to be critical w.r.t such variations becomes $\sum_{i=1}^{m} \phi_i^2 = 1$ in Ω. As the eigenfunctions of the Dirichlet Laplacian vanish on the boundary $\partial\Omega$, this last condition can never be fulfilled by functions of E_k. Thus, we have the following:

Proposition 3.1. *There is no critical domain (Ω, g) for the k-th eigenvalue of the Dirichlet Laplacian under conformal volume-preserving variations of the metric g.*

4. Applications to the trace of the heat kernel

This section deals with critical domains of the trace of the heat kernel under Dirichlet boundary condition.

Recall that the heat kernel H of (Ω, g) under the Dirichlet boundary condition is defined to be the solution of the following parabolic problem:

$$\begin{cases} (\frac{\partial}{\partial t} - \Delta_y) H(t, x, y) = 0 \\ H(0, x, y) = \delta_x \\ \forall y \in \partial\Omega, \; H(t, x, y) = 0 \end{cases}$$

Its trace is the function

$$Y(t) = \int_\Omega H(t, x, x) v_g$$

The relationship between this kernel and the spectrum of the Dirichlet Laplacian is given by

$$H(t, x, y) = \sum_{k \geq 1} e^{-\lambda_k t} \phi_k(x) \phi_k(y)$$

where $(\phi)_{k \geq 1}$ is an $L^2(\Omega, g)$-orthonormal family of eigenfunctions satisfying

$$\begin{cases} \Delta\phi_k = \lambda_k\phi_k \text{ in } \Omega \\ \phi_k = 0 \text{ on } \partial\Omega \end{cases}$$

and then,

$$Y(t) = \sum_{k\geq 1} e^{-\lambda_k t}. \tag{8}$$

Let Ω_ε be a smooth deformation of Ω and let $Y_\varepsilon(t) = \sum_{k\geq 1} e^{-\lambda_{k,\varepsilon} t}$ be the corresponding heat trace function. Unlike the eigenvalues, the function $Y_\varepsilon(t)$ is always differentiable in ε and *the domain Ω will be said critical for the trace of the heat kernel under the Dirichlet boundary condition at time t if, for any volume-preserving deformation Ω_ε of Ω, we have*

$$\frac{d}{d\varepsilon} Y_\varepsilon(t)\big|_{\varepsilon=0} = 0$$

From the results of section 3 above, one can deduce the variation formula for the heat trace. For this, we need to introduce the mixed second derivative $d_S H(t)|_x$ of H at the point x defined as the smooth 2-tensor given by

$$d_S H(t)|_x(X, X) = \frac{\partial^2}{\partial\alpha\partial\beta} H(t, c(\alpha), c(\beta))\big|_{\alpha=\beta=0},$$

where c is a curve in Ω such that $c(0) = x$ and $\dot{c}(0) = X$. It is easy to check that

$$d_S H(t) = \sum_{k\geq 1} e^{-\lambda_k t} d\phi_k \otimes d\phi_k$$

Theorem 4.1. *Let $\Omega_\varepsilon = f_\varepsilon(\Omega)$ be a volume-preserving deformation of Ω. We have, $\forall t > 0$,*

$$\frac{d}{d\varepsilon} Y_\varepsilon(t)\big|_{\varepsilon=0} = -t \int_{\partial\Omega} v\, d_S H(t)(\nu, \nu) v_{\bar{g}} = \frac{t}{2} \int_{\partial\Omega} v\, \Delta H(t, x, x) v_{\bar{g}}$$

where $v = g(\frac{d}{d\varepsilon} f_\varepsilon\big|_{\varepsilon=0}, \nu)$.

Proof. The proof can be derived from the first variation formula of the heat kernel that can be found in the paper of Ray and Singer [28, Proposition 6.1]. Nevertheless, at least in the case where the ambient manifold is real analytic, the formula of Theorem 4.1 can be obtained as an immediate consequence of Hadamard's type formula of Section 2, thanks to the relation (8) above. Indeed, in this manner we obtain, $\forall t > 0$,

$$\frac{d}{d\varepsilon} Y_\varepsilon(t)\big|_{\varepsilon=0} = -t \sum_{k\geq 1} e^{-\lambda_k t} \int_{\partial\Omega} v \left(\frac{\partial\phi_k}{\partial\nu}\right)^2 v_{\bar{g}}$$

where (λ_k, ϕ_k) are as above. To get the desired formula for $Y_\varepsilon(t)$ it suffices to notice that

$$d_S H(t)(\nu,\nu) = \sum_{k\geq 1} e^{-\lambda_k t} d\phi_k \otimes d\phi_k(\nu,\nu) = \sum_{k\geq 1} e^{-\lambda_k t}\left(\frac{\partial \phi_k}{\partial \nu}\right)^2 .$$

□

An immediate consequence is the following

Corollary 4.1. *The following conditions are equivalent:*

(i) *The domain Ω is critical for the trace of the Dirichlet heat kernel at the time t under volume-preserving domain deformations,*
(ii) *$\Delta H(t,x,x)$ is constant on the boundary $\partial\Omega$,*
(iii) *for any positive integer k and any $L^2(\Omega, g)$-orthonormal basis $\phi_1, \cdots \phi_p$ of the eigenspace E_k of λ_k, $\sum_{i\leq p}\left(\frac{\partial\phi_i}{\partial\nu}\right)^2$ is constant on $\partial\Omega$.*

Recall that if ρ is an isometry of (Ω, g), then, $\forall x \in \Omega$ and $\forall t > 0$, $H(t, \rho(x), \rho(x)) = H(t, x, x)$. In particular, if Ω is a ball of $\mathbb{R}^n$ endowed with a rotationally symmetric Riemannian metric g given in polar coordinates by $g = a^2(r)dr^2 + b^2(r)d\sigma^2$, where $d\sigma^2$ is the standard metric of the unit sphere $\mathbb{S}^{n-1}$, then $H(t,x,x)$ should be radial (that is depend only on the parameter r). Therefore, the function $\Delta H(t,x,x)$ is also radial and then it is constant on the boundary of the ball.

Corollary 4.2. *Let g be a a rotationally symmetric Riemannian metric on $\mathbb{R}^n$. The geodesic balls centered at the origin are critical domains for the trace of the Dirichlet heat kernel under volume-preserving domain deformations.*

In particular, geodesic balls of Riemannian space forms are critical for the trace of the Dirichlet heat kernel under volume-preserving domain deformations.

The Minakshisundaram-Pleijel asymptotic expansion of the trace of the heat kernel can also informs us about the geometric properties of extremal or critical domains. Indeed, it is well known that there exists a sequence $(a_i)_{i\in\mathbb{N}}$ of real numbers such that for sufficiently small $t > 0$, we have:

$$Y(t) = (4\pi t)^{\frac{-n}{2}} \sum_{k\geq 0} a_k t^{\frac{k}{2}}$$

with (see for instance [6, 7]):

$$a_0 = vol(\Omega, g),$$

$$a_1 = -\frac{\sqrt{\pi}}{2} vol(\partial\Omega, \bar{g}),$$

$$a_2 = \frac{1}{6}\left\{\int_\Omega scal_g v_g + 2\int_{\partial\Omega} tr A\, v_{\bar{g}}\right\},$$

$$a_3 = \frac{\sqrt{\pi}}{192}\left\{\int_{\partial\Omega}\left(-16scal_g - 7(trA)^2 + 10|A|^2 + 8\rho_g(\nu,\nu)\right)v_{\bar g}\right\},$$

where $scal_g$ and ρ_g are respectively the scalar and the Ricci curvatures of (Ω, g), A is the shape operator of the boundary $\partial\Omega$ (i.e $\forall X \in T\partial\Omega$, $A(X) = D_X\nu$) and trA is the trace of A (i.e $(n-1)$-times the mean curvature of $\partial\Omega$).

An immediate consequence of these formulae is the following: Suppose that for any domain Ω' having the same volume as Ω, we have $Y_{\Omega'}(t) \leq Y_\Omega(t), \forall t > 0$, then $vol\,\partial\Omega' \geq vol\,\partial\Omega$. Consequently, we have

Proposition 4.1. *If the domain Ω maximizes Y at any time $t > 0$ among all the domains of the same volume, then Ω is a solution of the isoperimetric problem in (M, g), that is, $\forall\Omega' \subset M$ such that $vol\Omega = vol\Omega'$, we have $vol\partial\Omega' \geq vol\partial\Omega$.*

Another consequence of the Minakshisundaram-Pleijel asymptotic expansion is the following

Theorem 4.2. *If the domain Ω is a critical domain of the trace of the Dirichlet heat kernel at any time $t > 0$, then $\partial\Omega$ has constant mean curvature. If in addition the Ricci curvature (resp. the sectional curvature) of the ambient space (M, g) is constant in a neighborhood of Ω, then $tr(A^2)$ (resp. $tr(A^3)$) is constant on $\partial\Omega$.*

Proof. Let $\Omega_\varepsilon = f_\varepsilon(\Omega)$ be a volume-preserving variation of Ω and let us denote for any ε by $(a_{i,\varepsilon})_{i\geq 0}$ the coefficients of the asymptotic expansions of $Y_\varepsilon(t)$. Since $\frac{d}{d\varepsilon}Y_\varepsilon(t)\big|_{\varepsilon=0} = 0$, we have for any $i \geq 0$, $\frac{d}{d\varepsilon}a_{i,\varepsilon}\big|_{\varepsilon=0} = 0$ (see for instance [16] for an analytic justification for this last assertion). In particular, $\frac{d}{d\varepsilon}vol(\partial\Omega_\varepsilon)\big|_{\varepsilon=0} = 0$ for any volume-preserving variation of Ω. This property is known to be equivalent to the fact that the mean curvature of $\partial\Omega$ is constant (see for instance [29]).

Now, let us suppose that the Ricci curvature of (M, g) is constant in a neighborhood of Ω, then for any small ε, we have:

$$\begin{aligned} a_{2,\varepsilon} &= \frac{1}{6}\left\{scal_g vol(\Omega_\varepsilon) + 2\int_{\partial\Omega_\varepsilon}(trA_\varepsilon)\,v_{\bar g}\right\} \\ &= \frac{1}{6}\left\{scal_g vol(\Omega) + 2\int_{\partial\Omega_\varepsilon}(trA_\varepsilon)\,v_{\bar g}.\right\} \end{aligned}$$

Hence, we have (see for instance [29]):

$$\begin{aligned} \frac{d}{d\varepsilon}\int_{\partial\Omega_\varepsilon}(trA_\varepsilon)\,v_{\bar g}\big|_{\varepsilon=0} &= \int_{\partial\Omega}\left(\Delta_{\bar g}v - \rho(\nu,\nu)v - (trA^2)v\right)v_{\bar g} \\ &\quad + \frac{1}{2}\int_{\partial\Omega} trA\left(div_{\bar g}V^T + v\,trA\right)v_{\bar g}, \end{aligned}$$

where $V = \frac{df_\varepsilon}{d\varepsilon}\big|_{\varepsilon=0} = v\,\nu + V^T$ on the boundary $\partial\Omega$.
Since $\int_{\partial\Omega} v\, v_{\bar g} = 0$ and trA and $\rho(\nu,\nu)$ are constant on $\partial\Omega$, we have:

$$\frac{d}{d\varepsilon} a_{2,\varepsilon}\big|_{\varepsilon=0} = \frac{1}{3}\int_{\partial\Omega} (trA^2) v\, v_{\bar g} = 0.$$

It follows that trA^2 is constant on $\partial\Omega$.

As before, we have

$$\frac{d}{d\varepsilon} a_{3,\varepsilon}\big|_{\varepsilon=0} = \frac{\sqrt{\pi}}{192}\left(-7\,\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\int_{\partial\Omega_\varepsilon} (trA_\varepsilon)^2\, v_{\bar g} + 10\,\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\int_{\partial\Omega_\varepsilon} trA_\varepsilon^2\, v_{\bar g}\right)$$

but,

$$\begin{aligned}\frac{d}{d\varepsilon}\int_{\partial\Omega_\varepsilon} (trA_\varepsilon)^2\, v_{\bar g}\big|_{\varepsilon=0} &= 2\int_{\partial\Omega} trA\left(\Delta_{\bar g} v - \rho(\nu,\nu)v - (trA^2)v\right) v_{\bar g} \\ &+ \frac{1}{2}\int_{\partial\Omega} (trA)^2\left(div_{\bar g}V^T + v\, trA\right) v_{\bar g} \\ &= 0\end{aligned}$$

since trA, trA^2 and $\rho(\nu,\nu)$ are constants. Thus,

$$\frac{d}{d\varepsilon} a_{3,\varepsilon}\big|_{\varepsilon=0} = \frac{10\sqrt{\pi}}{192}\,\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\int_{\partial\Omega_\varepsilon} trA_\varepsilon^2\, v_{\bar g}.$$

After some straightforward but long computations we obtain, using the fact that the sectional curvature is constant in a neighborhood of Ω, and that trA and trA^2 are constant,

$$\frac{d}{d\varepsilon} a_{3,\varepsilon}\big|_{\varepsilon=0} = c\int_{\partial\Omega} trA^3\, v\, v_{\bar g} = 0,$$

where c is a constant. This proves that trA^3 is constant. □

Alexandrov's Theorem [1] shows that in the Euclidean space, the geodesic spheres are the only embedded compact hypersurfaces of constant mean curvature. This theorem was extended to hypersurfaces of the hyperbolic space and the standard hemisphere(see [24]). Since the boundary of a critical domain of the trace of the heat kernel is an embedded hypersurface of constant mean curvature, we have the

Corollary 4.3. *Let (M,g) be one of the following spaces:*

- *The Euclidean space.*
- *The Hyperbolic space.*
- *The standard Hemisphere.*

Then a domain Ω of (M,g) is critical for the trace of the Dirichlet heat kernel if and only if Ω is a geodesic ball.

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